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A Decreasing Sequence of Eigenvalue Localization Regions*

Oscar Rojo and Ricardo L. Soto
Universidad Católica del Norte
Antofagasta, Chile

and

Héctor Rojo
Universidad de Antofagasta
Antofagasta, Chile

Submitted by Richard A. Brualdi

ABSTRACT

We construct a decreasing sequence of rectangles (R_p) in such a way that all the eigenvalues of a complex matrix A are contained in each rectangle. When A is a matrix with real spectrum or a normal matrix, each R_p can be obtained without knowing the eigenvalues of A.

1. INTRODUCTION

Bounds for the eigenvalues of a matrix have been obtained by many authors. In general, these bounds are given in terms of a simple function of the entries of the matrix or the entries of one or more related matrices [4]. Bounds for eigenvalues using traces have been obtained in [6–10]. In [6] we proved that the eigenvalues λ_j of a complex matrix A of order n lie in the

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disk

$$\left|\lambda - \frac{\operatorname{tr} A}{n}\right| \leqslant \left[\frac{n-1}{n}\left(\sum_{i=1}^{n}|\lambda_i|^2 - \frac{|\operatorname{tr} A|^2}{n}\right)\right]^{1/2}. \tag{1.1}$$

We observe that if the eigenvalues of A are all real, then from (1.1), we have

$$\left|\lambda_j - \frac{\operatorname{tr} A}{n}\right| \leqslant \left[\frac{n-1}{n}\left(\operatorname{tr}(A^2) - \frac{(\operatorname{tr} A)^2}{n}\right)\right]^{1/2} \tag{1.2}$$

for all j. This result was given in [9, Theorem 2.1].

In [7] we proved that the eigenvalues of a complex matrix A of order n lie in the rectangle

$$\left[\frac{\operatorname{Re}(\operatorname{tr} A)}{n} - \alpha, \frac{\operatorname{Re}(\operatorname{tr} A)}{n} + \alpha\right] \times \left[\frac{\operatorname{Im}(\operatorname{tr} A)}{n} - \beta, \frac{\operatorname{Im}(\operatorname{tr} A)}{n} + \beta\right], \tag{1.3}$$

where

$$\alpha = \left[\frac{n-1}{n}\left(\sum_{k=1}^{n}(\operatorname{Re}\lambda_k)^2 - \frac{[\operatorname{Re}(\operatorname{tr} A)]^2}{n}\right)\right]^{1/2}, \tag{1.4}$$

$$\beta = \left[\frac{n-1}{n}\left(\sum_{k=1}^{n}(\operatorname{Im}\lambda_k)^2 - \frac{[\operatorname{Im}(\operatorname{tr} A)]^2}{n}\right)\right]^{1/2}. \tag{1.5}$$

This rectangle is included in the disk (1.1), and its vertices are on the boundary circle of that disk.

Upper bounds for the expressions occurring in (1.1), (1.4), and (1.5) can be obtained using Schur's inequalities [3] or measures of nonnormality [1, 3].

In Section 2 we construct a decreasing sequence of rectangles (R_p) in such a way that all the eigenvalues λ_j of a complex matrix A of order $n + 1$

are contained in each R_p. That is,

$$\left|\operatorname{Re}\lambda_k - \frac{\operatorname{Re}(\operatorname{tr} A)}{n+1}\right| \leqslant \alpha_p \quad \text{and} \quad \left|\operatorname{Im}\lambda_k - \frac{\operatorname{Im}(\operatorname{tr} A)}{n+1}\right| \leqslant \beta_p$$

for $k = 1, \ldots, n+1$, $p \geqslant 0$, where α_p and β_p will be defined later. It turns out that R_1 is the rectangle given in (1.3). When A is a matrix with real spectrum or a normal matrix, each rectangle R_p can be computed without knowing the eigenvalues of A.

2. A DECREASING SEQUENCE OF RECTANGLES CONTAINING THE EIGENVALUES OF A MATRIX

The eigenvalue localization region (1.3) was obtained by the use of the following lemma which we proved in [5].

LEMMA 2.1. *If $d_1, \ldots, d_n$ are real numbers such that $d_1 + \cdots + d_n = 0$, then for all k*

$$d_k^2 \leqslant \frac{n-1}{n} \sum_{i=1}^{n} d_i^2. \tag{2.1}$$

We extend the result of Lemma 2.1 to any even exponent.

LEMMA 2.2. *If $p \in \mathbb{N}$ and $x_1, \ldots, x_n \in \mathbb{R}$ are such that $x_1 + x_2 + \cdots + x_n = 1$, then*

$$\frac{n^{2p-1}+1}{n^{2p-1}} \leqslant \sum_{i=1}^{n} x_i^{2p} + 1. \tag{2.2}$$

Proof. We define

$$f, h : \mathbb{R}^n \to \mathbb{R}$$

by

$$f(x_1, \ldots, x_n) = \sum_{i=1}^{n} x_i^{2p} + 1,$$

$$h(x_1, \ldots, x_n) = \sum_{i=1}^{n} x_i - 1,$$

and

$$L(x_1, \ldots, x_n, \lambda) = f(x_1, \ldots, x_n) + \lambda h(x_1, \ldots, x_n), \qquad \lambda \in \mathbb{R}.$$

We easily see that $x_1^* = x_2^* = \cdots = x_n^* = 1/n$ is the unique solution of the equation $\nabla L(\mathbf{x}, \lambda) = 0$. Moreover, we have

$$\frac{\partial^2 L}{\partial x_k^2} = 2p(2p-1)x_k^{2p-2}, \qquad k = 1, \ldots, n,$$

and

$$\frac{\partial^2 L}{\partial x_k \, \partial x_j} = 0, \qquad k, j = 1, \ldots, n, \quad k \neq j.$$

Hence,

$$\nabla^2 L(\mathbf{x}^*, \lambda) = \frac{2p(2p-1)}{n^{2p-2}} I_n$$

is a positive definite matrix, where I_n denotes the identity matrix of order n. Therefore, f attains a minimum value at the point $\mathbf{x}^* = (1/n, \ldots, 1/n)$, and this minimum value is $(n^{2p-1}+1)/n^{2p-1}$. ∎

THEOREM 2.3. *If $d_1, \ldots, d_{n+1}$ are real numbers such that $d_1 + \cdots + d_{n+1} = 0$, then*

$$d_k^{2p} \leqslant \frac{n^{2p-1}}{n^{2p-1}+1} \sum_{i=1}^{n+1} d_i^{2p} \tag{2.3}$$

for $k = 1, \ldots, n+1$ and $p \in \mathbb{N}$.

Proof. If $d_k = 0$, then there is nothing to prove. Let $d_k \neq 0$ and $x_i = -d_i/d_k$ for $i \neq k$. Hence $\sum_{i \neq k} x_i = 1$. Thus by Lemma 2.2 we have

$$\frac{n^{2p-1}+1}{n^{2p-1}} \leqslant \sum_{i \neq k} x_i^{2p} + 1,$$

and the result follows. ∎

We observe that for $p = 1$, we have the inequality of Lemma 2.1. If $d_k = 1$ and $d_i = -1/n$ for $i \neq k$, then we have an equality in (2.3), which shows that the inequality is sharp.

THEOREM 2.4. *Let A be a complex matrix of order $n+1$ with eigenvalues $\lambda_1, \dots, \lambda_{n+1}$. Then all the λ_k ($k = 1, \dots, n+1$) lie in the rectangle*

$$\left[\frac{\operatorname{Re}(\operatorname{tr} A)}{n+1} - \alpha_p, \frac{\operatorname{Re}(\operatorname{tr} A)}{n+1} + \alpha_p\right] \times \left[\frac{\operatorname{Im}(\operatorname{tr} A)}{n+1} - \beta_p, \frac{\operatorname{Im}(\operatorname{tr} A)}{n+1} + \beta_p\right], \tag{2.4}$$

where $p \in \mathbb{N}$ and

$$\alpha_p = \left[\frac{n^{2p-1}}{n^{2p-1}+1} \sum_{i=1}^{n+1} \left(\operatorname{Re} \lambda_i - \frac{\operatorname{Re}(\operatorname{tr} A)}{n+1}\right)^{2p}\right]^{1/2p}, \tag{2.5}$$

$$\beta_p = \left[\frac{n^{2p-1}}{n^{2p-1}+1} \sum_{i=1}^{n+1} \left(\operatorname{Im} \lambda_i - \frac{\operatorname{Im}(\operatorname{tr} A)}{n+1}\right)^{2p}\right]^{1/2p}. \tag{2.6}$$

That is,

$$\left|\operatorname{Re} \lambda_j - \frac{\operatorname{Re}(\operatorname{tr} A)}{n+1}\right| \leqslant \alpha_p \quad \textit{and} \quad \left|\operatorname{Im} \lambda_j - \frac{\operatorname{Im}(\operatorname{tr} A)}{n+1}\right| \leqslant \beta_p.$$

Proof. Since

$$\sum_{i=1}^{n+1} \left(\lambda_i - \frac{\operatorname{tr} A}{n+1}\right) = 0,$$

we have

$$\sum_{i=1}^{n+1} \left(\operatorname{Re} \lambda_i - \frac{\operatorname{Re}(\operatorname{tr} A)}{n+1}\right) = 0 \quad \text{and} \quad \sum_{i=1}^{n+1} \left(\operatorname{Im} \lambda_i - \frac{\operatorname{Im}(\operatorname{tr} A)}{n+1}\right) = 0.$$

Then it follows from Theorem 2.3 that

$$\left|\operatorname{Re} \lambda_k - \frac{\operatorname{Re}(\operatorname{tr} A)}{n+1}\right| \leqslant \alpha_p \quad \text{and} \quad \left|\operatorname{Im} \lambda_k - \frac{\operatorname{Im}(\operatorname{tr} A)}{n+1}\right| \leqslant \beta_p$$

for $k = 1, \dots, n+1$. Thus, each λ_k lies in the rectangle (2.4), $R_p = X_p \times Y_p$, with X_p and Y_p defined by

$$X_p = \left[\frac{\operatorname{Re}(\operatorname{tr} A)}{n+1} - \alpha_p, \frac{\operatorname{Re}(\operatorname{tr} A)}{n+1} + \alpha_p \right], \tag{2.7}$$

$$Y_p = \left[\frac{\operatorname{Im}(\operatorname{tr} A)}{n+1} - \beta_p, \frac{\operatorname{Im}(\operatorname{tr} A)}{n+1} + \beta_p \right]. \quad \blacksquare \tag{2.8}$$

COROLLARY 2.5. *Let A be a complex matrix of order $n+1$ with real eigenvalues $\lambda_1, \dots, \lambda_{n+1}$. Then all the λ_k lie in the interval*

$$\left[\frac{\operatorname{tr} A}{n+1} - \alpha_p, \frac{\operatorname{tr} A}{n+1} + \alpha_p \right],$$

where $p \in \mathbb{N}$,

$$\alpha_p = \left\{ \frac{n^{2p-1}}{n^{2p-1}+1} \operatorname{tr}\left[\left(A - \frac{\operatorname{tr} A}{n+1} \right)^{2p} \right] \right\}^{1/2p}, \tag{2.9}$$

and I is the identity matrix of order $n+1$.

Proof. From Theorem 2.4 we have

$$\alpha_p = \left[\frac{n^{2p-1}}{n^{2p-1}+1} \sum_{i=1}^{n+1} \left(\lambda_i - \frac{\operatorname{tr} A}{n+1} \right)^{2p} \right]^{1/2p}, \qquad \beta_p = 0,$$

and

$$\sum_{i=1}^{n+1} \left(\lambda_i - \frac{\operatorname{tr} A}{n+1} \right)^{2p} = \operatorname{tr}\left[\left(A - \frac{\operatorname{tr} A}{n+1} I \right)^{2p} \right]. \qquad \blacksquare$$

Next, we shall be concerned with the convergence of the sequences (X_p) and (Y_p) defined in the proof of Theorem 2.4. We shall actually prove that these sequences are decreasing, that is, $X_{p+1} \subseteq X_p$ and $Y_{p+1} \subseteq Y_p$. We need the following lemma:

LEMMA 2.6. *If $p \in \mathbb{N}$ and $x_1, \dots, x_n$ are real numbers such that $x_1 + x_2 + \cdots + x_n = 1$, then*

$$\frac{\left(n^{2p+1}+1\right)^p}{n\left(n^{2p-1}+1\right)^{p+1}} \geqslant \frac{\left(\sum_{i=1}^n x_i^{2p+2}+1\right)^p}{\left(\sum_{i=1}^n x_i^{2p}+1\right)^{p+1}}. \tag{2.10}$$

Proof. We define, for $\mathbf{x} = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$,

$$f(\mathbf{x}) = \frac{\left(\sum_{i=1}^n x_i^{2p+2}+1\right)^p}{\left(\sum_{i=1}^n x_i^{2p}+1\right)^{p+1}}, \qquad h(\mathbf{x}) = \sum_{i=1}^n x_i - 1,$$

and

$$L(\mathbf{x}) = f(\mathbf{x}) + \lambda h(\mathbf{x}), \qquad \lambda \in \mathbb{R}.$$

As in Lemma 2.2, one can easily prove that $\mathbf{x}^* = (1/n, 1/n, \dots, 1/n)$ is the unique solution of the condition $\nabla L(\mathbf{x}, \lambda) = 0$. We can prove that

$$\frac{\partial^2 L}{\partial x_k\, \partial x_j}(\mathbf{x}^*) - \frac{\partial^2 L}{\partial x_k^2}(\mathbf{x}^*) = \beta,$$

where

$$\beta = 2p(p+1)f(\mathbf{x}^*)n\frac{n^{2p+1}(2p-1) - n^{2p-1}(2p+1) - 2}{(1+n^{2p-1})(1+n^{2p+1})}.$$

The constant β is positive for $n > 2$ and for any natural p., For $n = 2$, β is positive for $p > 1$ and zero for $p = 1$. The Hessian matrix $\nabla^2 L(\mathbf{x}^*, \lambda)$ has all its diagonal entries equal to α, and all its nondiagonal entries equal to $\alpha + \beta$. Let $\mathbf{y} = (y_1, y_2, \dots, y_n)^T$ such that $\mathbf{y}^T(\nabla h(\mathbf{x}^*)) = 0$. It follows that $\sum_{i=1}^n y_i = 0$ and $\mathbf{y}^T \nabla^2 L(\mathbf{x}^*, \lambda)\mathbf{y} = -\beta \sum_{i=1}^n y_i^2$. Since β is positive, $\nabla^2 L(\mathbf{x}^*, \lambda)$ is a negative definite matrix and $f(\mathbf{x}^*) = (n^{2p+1}+1)^p / n(n^{2p-1}+1)^{p+1}$ is the maximum. ∎

REMARK. If $x_1, \dots, x_n$ are real numbers such that $x_1 + x_2 + \cdots + x_n = 1$ and $x_i \neq x_j$ for some $i \neq j$, then the inequality in (2.10) is strict except for $n = 2$ and $p = 1$. In this case equality holds.

Now we are ready to prove one of the main results of this section.

THEOREM 2.7. *If $d_1, d_2, d_3, \dots, d_{n+1}$ are real numbers such that $d_1 + d_2 + d_3 + \cdots + d_n + d_{n+1} = 0$ and $d_k \neq 0$ for some k, then the sequence (s_p) defined by*

$$s_p = \left(\frac{n^{2p-1}}{n^{2p-1}+1} \sum_{i=1}^{n+1} d_i^{2p} \right)^{1/2p}$$

is decreasing.

Proof. Let $r = s_{p+1}/s_p$; then

$$r^{2p(p+1)} = \frac{n(n^{2p-1}+1)^{p+1}}{(n^{2p+1}+1)^p} \frac{\left(\sum_{i=1}^{n+1} d_i^{2p+2}\right)^p}{\left(\sum_{i=1}^{n+1} d_i^{2p}\right)^{p+1}}.$$

Let $x_i = -d_i/d_k$ for $i \neq k$. Then $\sum_{i \neq k} x_i = 1$, and Lemma 2.6 can be applied to obtain

$$\frac{\left(\sum_{i=1}^{n+1} d_i^{2p+2}\right)^p}{\left(\sum_{i=1}^{n+1} d_i^{2p}\right)^{p+1}} \leqslant \frac{(n^{2p+1}+1)^p}{n(n^{2p-1}+1)^{p+1}}.$$

Hence,

$$r^{2p(p+1)} = \frac{n(n^{2p-1}+1)^{p+1}}{(n^{2p+1}+1)^p} \frac{\left(\sum_{i=1}^{n+1} d_i^{2p+2}\right)^p}{\left(\sum_{i=1}^{n+1} d_i^{2p}\right)^{p+1}} \leqslant 1,$$

and so $r = s_{p+1}/s_p \leqslant 1$. ■

We now prove that the sequence (R_p) is decreasing.

THEOREM 2.8. *Let A be a complex matrix of order $n+1$ with eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_{n+1}$. Then the sequence (R_p), $R_p = X_p \times Y_p$, of Theorem 2.4 is decreasing.*

Proof. Consider the sequence of intervals X_p. In order to show that $X_{p+1} \subseteq X_p$ it is sufficient to prove that $\alpha_{p+1} \leqslant \alpha_p$. The real numbers

$$\operatorname{Re} \lambda_1 - \frac{\operatorname{Re}(\operatorname{tr} A)}{n+1}, \qquad \operatorname{Re} \lambda_2 - \frac{\operatorname{Re}(\operatorname{tr} A)}{n+1}, \dots, \qquad \operatorname{Re} \lambda_{n+1} - \frac{\operatorname{Re}(\operatorname{tr} A)}{n+1}$$

satisfy the condition of Theorem 2.7. Then the sequence (α_p) defined by (2.5) is decreasing. A similar argument applied to the real numbers

$$\operatorname{Im} \lambda_1 - \frac{\operatorname{Im}(\operatorname{tr} A)}{n+1}, \qquad \operatorname{Im} \lambda_2 - \frac{\operatorname{Im}(\operatorname{tr} A)}{n+1}, \ldots,$$

$$\operatorname{Im} \lambda_{n+1} - \frac{\operatorname{Im}(\operatorname{tr} A)}{n+1},$$

proves that the sequence of intervals Y_p is also decreasing. ■

COROLLARY 2.9. *Let A be a complex matrix of order n + 1 with real spectrum. Let α_p be as in* (2.9). *Then the sequence of intervals* $[(\operatorname{tr} A)/(n+1) - \alpha_p, (\operatorname{tr} A)/(n+1) + \alpha_p]$, *containing the eigenvalues λ_k of A, is decreasing.*

Proof. It is an immediate consequence of Theorem 2.7 with $d_k = \lambda_k - (\operatorname{tr} A)/(n+1)$ for $k = 1, 2, \ldots, n+1$. ■

THEOREM 2.10. *Let A be a complex matrix of order n + 1 with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_{n+1}$. Then the sequences (α_p) and (β_p), defined by* (2.5) *and* (2.6), *have the following limits*:

$$\alpha = \lim_{p \to \infty} \alpha_p = \max_i \left| \operatorname{Re} \lambda_i - \frac{\operatorname{Re}(\operatorname{tr} A)}{n+1} \right|$$

$$\beta = \lim_{p \to \infty} \beta_p = \max_i \left| \operatorname{Im} \lambda_i - \frac{\operatorname{Im}(\operatorname{tr} A)}{n+1} \right|.$$

Proof. Let $\mathbf{x}$ be the vector with components

$$\lambda_1 - \frac{\operatorname{tr} A}{n+1}, \qquad \lambda_2 - \frac{\operatorname{tr} A}{n+1}, \ldots, \qquad \lambda_{n+1} - \frac{\operatorname{tr} A}{n+1}.$$

We write

$$\operatorname{Re}\mathbf{x} = \left[\operatorname{Re}\lambda_1 - \frac{\operatorname{Re}(\operatorname{tr} A)}{n+1}, \operatorname{Re}\lambda_2 - \frac{\operatorname{Re}(\operatorname{tr} A)}{n+1}, \ldots, \operatorname{Re}\lambda_{n+1} - \frac{\operatorname{Re}(\operatorname{tr} A)}{n+1}\right]^T,$$

$$\operatorname{Im}\mathbf{x} = \left[\operatorname{Im}\lambda_1 - \frac{\operatorname{Im}(\operatorname{tr} A)}{n+1}, \operatorname{Im}\lambda_2 - \frac{\operatorname{Im}(\operatorname{tr} A)}{n+1}, \ldots, \operatorname{Im}\lambda_{n+1} - \frac{\operatorname{Im}(\operatorname{tr} A)}{n+1}\right]^T.$$

From (2.5) and (2.6), we have

$$\alpha_p = \left[\frac{n^{2p-1}}{n^{2p-1}+1}\right]^{1/2p} \|\operatorname{Re}\mathbf{x}\|_{2p} \quad \text{and} \quad \beta_p = \left[\frac{n^{2p-1}}{n^{2p-1}+1}\right]^{1/2p} \|\operatorname{Im}\mathbf{x}\|_{2p},$$

where $\|\mathbf{v}\|_{2p}$ denotes the $2p$-norm of the vector $\mathbf{v}$. Since $\|\mathbf{v}\|_{2p}$ tends to $\|\mathbf{v}\|_\infty$ and $[n^{2p-1}/(n^{2p-1}+1)]^{1/2p}$ tends to 1 as p tends to infinity, it is clear that

$$\alpha = \lim_{p\to\infty}\alpha_p = \|\operatorname{Re}\mathbf{x}\|_\infty = \max_i\left|\operatorname{Re}\lambda_i - \frac{\operatorname{Re}(\operatorname{tr} A)}{n+1}\right|,$$

$$\beta = \lim_{p\to\infty}\beta_p = \|\operatorname{Im}\mathbf{x}\|_\infty = \max_i\left|\operatorname{Im}\lambda_i - \frac{\operatorname{Im}(\operatorname{tr} A)}{n+1}\right|.$$

If in addition all the eigenvalues of A are real numbers,

$$\alpha = \max_{1\leqslant i\leqslant n+1}\left|\lambda_i - \frac{\operatorname{tr} A}{n+1}\right| \quad \text{and} \quad \beta = 0. \qquad \blacksquare$$

REMARK.

(a) Theorem 2.10 tells us that the sequence of rectangles R_p defined by (2.4) is convergent to the smallest rectangle centered at $(\operatorname{tr} A)/(n+1)$ and containing all the eigenvalues of the matrix A.

(b) The disk centered at $(\mathrm{tr}\, A)/(n+1)$ and with radius $\sqrt{\alpha^2+\beta^2}$ is the smallest disk with that center containing all the eigenvalues of A.

Our purpose now is to compute the intervals X_p and Y_p. For this, it is convenient to work with the matrix

$$B = A - \frac{\mathrm{tr}\, A}{n+1} I$$

instead of the matrix A, where I denotes the identity matrix of order $n+1$. Let $\lambda_1, \lambda_2, \ldots, \lambda_{n+1}$ be the eigenvalues of A. Then the eigenvalues of B are the numbers

$$\mu_1 = \lambda_1 - \frac{\mathrm{tr}\, A}{n+1}, \qquad \mu_2 = \lambda_2 - \frac{\mathrm{tr}\, A}{n+1}, \ldots,$$

$$\mu_{n+1} = \lambda_{n+1} - \frac{\mathrm{tr}\, A}{n+1}.$$

From (2.5) and (2.6), we have for the matrix B

$$\alpha_p = \left[\frac{n^{2p-1}}{n^{2p-1}+1} \sum_{i=1}^{n+1} (\mathrm{Re}\, \mu_i)^{2p}\right]^{1/2p}, \tag{2.11}$$

$$\beta_p = \left[\frac{n^{2p-1}}{n^{2p-1}+1} \sum_{i=1}^{n+1} (\mathrm{Im}\, \mu_i)^{2p}\right]^{1/2p}. \tag{2.12}$$

Let us define the matrices $C = \frac{1}{2}(B+B^*)$ and $D = (1/2i)(B - B^*)$. We consider the following cases:

Case 1: The Spectrum of A is real. If A has a real spectrum, then the spectrum of B is also a real set. From Corollary 2.5, we have

$$\alpha_p = \left(\frac{n^{2p-1}}{n^{2p-1}+1} \mathrm{tr}\, B^{2p}\right)^{1/2p} \quad \text{and} \quad \beta_p = 0 \quad \text{for all } p. \tag{2.13}$$

This expression for α_p allows us to obtain the real interval

$$X_p = \left[\frac{\operatorname{tr} A}{n+1} - \alpha_p, \frac{\operatorname{tr} A}{n+1} + \alpha_p \right]$$

without knowing the eigenvalues of A. We observe that in this case $(\operatorname{tr} A)/(n+1) - \alpha$ or $(\operatorname{tr} A)/(n+1) + \alpha$ is an eigenvalue of A.

Case 2: A is a normal matrix. If A is a normal matrix, then B is also a normal matrix. The spectra of C and D are

$$\{\operatorname{Re} \mu_1, \operatorname{Re} \mu_2, \dots, \operatorname{Re} \mu_{n+1}\} \quad \text{and} \quad \{\operatorname{Im} \mu_1, \operatorname{Im} \mu_2, \dots, \operatorname{Im} \mu_{n+1}\},$$

respectively. From (2.11) and (2.12), we have

$$\alpha_p = \left(\frac{n^{2p-1}}{n^{2p-1}+1} \operatorname{tr} C^{2p} \right)^{1/2p}, \tag{2.14}$$

$$\beta_p = \left(\frac{n^{2p-1}}{n^{2p-1}+1} \operatorname{tr} D^{2p} \right)^{1/2p}. \tag{2.15}$$

From (2.14) and (2.15), we see that each $R_p = X_p \times Y_p$ can be obtaining without knowing the eigenvalues of the normal matrix A.

3. EXAMPLES

EXAMPLE 3.1. Let us consider the matrix

$$A = \begin{bmatrix} 4 & 1 & 1 & 2 & 2 \\ 1 & 5 & 1 & 1 & 1 \\ 1 & 1 & 6 & 1 & 1 \\ 2 & 1 & 1 & 7 & 1 \\ 2 & 1 & 1 & 1 & 8 \end{bmatrix},$$

which was taken from the paper of Wolkowicz and Styan [9, Example 5]. Here the authors obtain, among others results, that are interval [0.203, 11.797] contains all the eigenvalues of A. We are in case 1. From (2.13) we obtain

p	α_p	$(\mathrm{tr}\, A)/n - \alpha_p$	$(\mathrm{tr}\, A)/n + \alpha_p$
1	5.7966	0.2034	11.7966
2	5.3975	0.6025	11.3975
4	5.1951	0.8049	11.1951
8	5.1718	0.8282	11.1778
16	5.1713	0.8287	11.1713
32	5.1713	0.8287	11.1713

The eigenvalues of A are 2.5711, 4.2961, 5.4346, 6.5268, and 11.1713.

EXAMPLE 3.2. Let A be the circulant matrix

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \\ 3 & 4 & 1 & 2 \\ 2 & 3 & 4 & 1 \end{bmatrix}.$$

This matrix is normal. Then we can use the results of case 2. We have $\alpha_p = 9$ for all p, and then the real parts of the eigenvalues are included in the interval $[-8, 10]$. Since A is a real matrix, the imaginary parts of the eigenvalues are included in the intervals $[-\beta_p, \beta_p]$. From (2.15), we obtain

p	β_p
1	2.4495
16	2.0438
256	2.0014

The eigenvalues of A are -2, 10, $-2 + 2i$, and $-2 - 2i$.

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